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METHODS FOR MEASUREMENT OF AVERAGE VALUES OF ENERGY AND MOMENTUM OF β-RECOILS

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Introduction.

The recoil in β-decay has been studied experimentally in various ways [(1—2)]; in all cases it turned out that considerable difficulties are involved in the measurements of energy and momentum of recoil particles. A major problem is always to obtain an ideal geometry for the instruments. Since it is desirable to use one-atomic gases as sources of the radioactivity, the source extends all over the apparatus. This makes it difficult to obtain an ideal geometry, i. e., to construct instruments which select particles in a narrow energyinterval only, giving directly the energy spectrum. For this reason the interpretation of the experimental results may introduce errors, and so far one has been unable to decide experimentally between the different possibilities as regards the angular correlation between β-particle and neutrino. It is thus of interest to look for experiments which allow of a more direct interpretation.

It is the purpose of this article to discuss some methods for the determination of average values of energy and momentum of the recoil particles in β-decay. It will be our aim to avoid measuring the energy distribution itself by means of an apparatus with complicated geometry. Instead, we want to carry out measurements of a few average values which can be obtained from instruments with a perfect geometry and while they do not give all details of the energy distribution are sufficient to decide between the different possibilities for the angular correlation between β-particle and neutrino.

The discussion given below indicates that it is possible to carry through this programme using rather simple instruments.

Measurements in Simple Fields.

Electric Fields.

The method of measuring the average value of the kinetic energy E_R divided by the charge z of the recoil particles[1] from β-radioactive noble gases which are followed by radioactive daughter substances has already been described elsewhere and applied to two radioactive Krypton isotopes, Kr^{88} and Kr^{89} (2). The method consists simply in counting the number of particles reaching the plates of a plane parallel condenser (see Fig. 1).

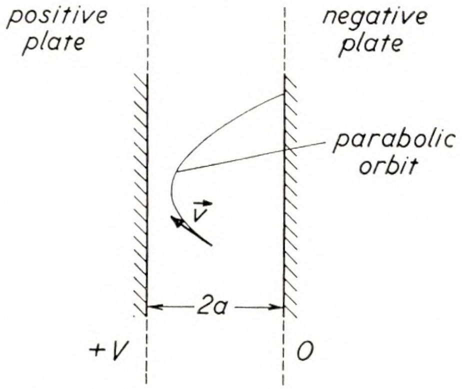

Fig. 1. The condenser used in the average energy measurements.

The radioactive gas is admitted into the space between the plates of the condenser at a pressure so low that the mean free path is considerably greater than the distance 2 a between the plates. A potential difference, V, is set up between the condenser plates, and the particles move in parabolic orbits. Let the number of recoil particles reaching the positive plate be N_+, and those reaching the negative plate be N_-. We consider the limit of very large condenser-plates. It is then easily shown that

$$\langle \frac{E_R}{z} \rangle = 6\, e V \frac{N_+}{N}, \tag{1}$$

[1] z is expected to be at least 10 % larger than unity because of several effects. A discussion of this question is being prepared by Mr. Aa. Winther. The most important effect comes from the change of the nuclear charge from Z to $Z + 1$ during the β-process.

where $\langle\,\rangle$ denotes the average value. The formula (1) will only hold if V fulfills the condition

$$eV \gg E_R^{\max}. \tag{2}$$

In (1) $N = N_+ + N_-$ is the total number of disintegrations. The counting of the numbers N_+ and N_- or rather the determination of the ratio of these two numbers which is sufficient for the determination of (1) is easily carried through by the tracer method because the daughter substances are radioactive.

We consider next the case that the condition (2) is not fulfilled. Let us assume that our recoil spectrum is a single line for which the energy $E_R = E_R^{\max} = \langle E_R \rangle$ and the charge $z = 1$. We find

$$N_+/N = \frac{1}{2}\left(1 - \frac{2}{3}\sqrt{\frac{eV}{E_R}}\right). \tag{3}$$

For $eV = E_R$ (1) and (3) join and have a common tangent. Thus if (1) is used for eV-values slightly below E_R the difference between (3) and (1) is of the second order in $E_R - eV = \Delta$. We find

$$\Delta N_+/N_+ = \frac{N_+(3) - N_+(1)}{N_+(3)} = -\frac{3}{4}\left(\frac{\Delta}{E_R}\right)^2. \tag{4}$$

In Fig. 2 is shown N_+/N and $6V\,N_+/N$ as functions of V. It is seen that for $Ve > E_R^{\max}$ the function $6V\,N_+/N$ is a constant. For $V = 0$ the functions have the values $^1/_2$ and 0 respectively. If we have a spectrum of recoil energies $P(E_R)\,dE_R$ instead of a single line we find for voltages below the maximum energy of the recoil spectrum

$$n_F(V) = N_+/N = \int_0^{eV} \frac{E_R}{6\,eV} P(E_R)\,dE_R + \int_{eV}^{E_R^{\max}} \left(\frac{1}{2} - \frac{1}{3}\sqrt{\frac{eV}{E_R}}\right) P(E_R)\,dE_R. \tag{5}$$

Complete knowledge of N_+/N for all values of V permits in principle a determination of $P(E_R)$, and one finds

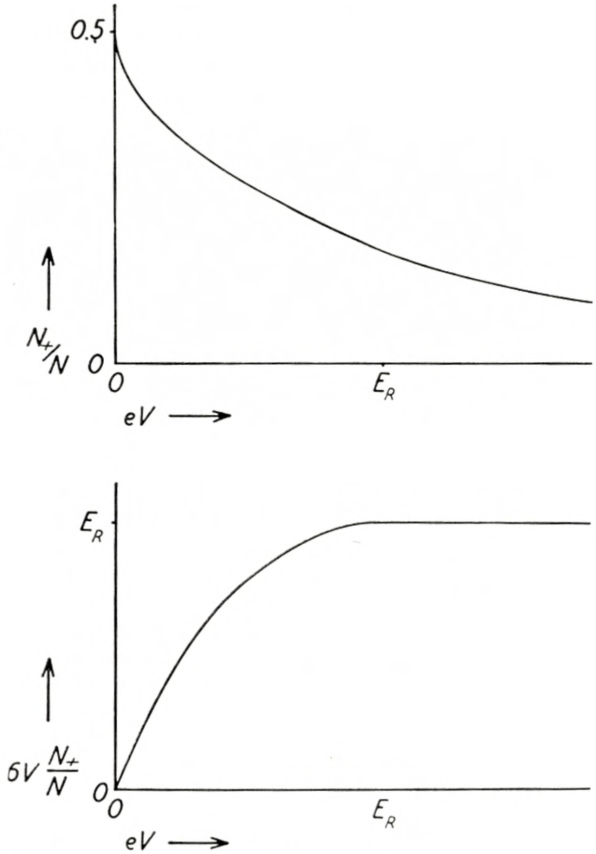

Fig. 2. N_+/N and $6VN_+/N$ as functions of V for a single line of energy E_R.

$$P(eV) = 4\frac{d}{d(eV)}n_F + 14\,eV\frac{d^2}{d(eV)^2}n_F + 4\,(eV)^2\frac{d^3}{d(eV)^3}n_F. \tag{6}$$

This relation shows that there is a one to one correspondance between n_F and P. Unfortunately, n_F is rather insensitive to the shape of the energy distribution. As an example let us consider a very simple distribution $P(E_R)$ i. e.

$$P(E_R) = \left\{\begin{array}{lll} 1/E_R^{\max} = \text{const.} & \text{for} & E_R \leqslant E_R^{\max} \\ 0 & \text{for} & E_R > E_R^{\max}. \end{array}\right\} \tag{7}$$

Obviously the value $1/E_R^{\max}$ accounts for the normalisation. The distribution (7) is shown in Fig. 3. From formula (5) we find

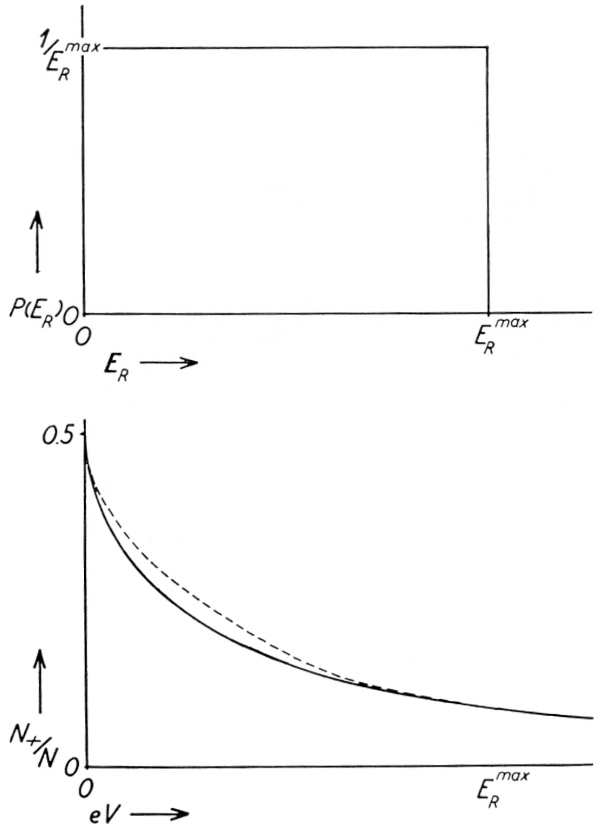

Fig. 3. The figure shows a hypotetical recoil energy-distribution together with the corresponding N_+/N curve. The dotted line is the N_+/N curve for a single line spectrum with the same average energy as the hypotetical distribution.

$$N_+/N = \left\{ \begin{array}{ll} \frac{1}{2} + \frac{1}{4} eV/E_R^{\max} - \frac{2}{3}\sqrt{\frac{eV}{E_R^{\max}}} & \text{for } eV \ll E_R^{\max} \\ \frac{1}{12} E_R^{\max}/eV & \text{for } eV \gg E_R^{\max}. \end{array} \right\} \quad (8)$$

Furthermore it is easily verified that (8) leads to (7) when formula (6) is applied. (8) is also shown in Fig. 10 together with the curve for a single line with the average energy $E_R^{\max}/2$. The small deviation between the curves illustrates the difficulties in carrying through an analysis of curves of the type N_+/N in order to obtain the energy spectrum, and it is seen that counting experiments are hardly sufficiently accurate for this purpose.

The method described so far uses a determination of the number of recoil particles reaching the condenser plates. This type of measurements is most easily carried out in such cases where the radioactive noble gas has a radioactive daughter substance as in the case of Kr^{88} or Kr^{89}. For these gases, however, several difficulties arise. First of all it is not easy to determine the disintegration schemes in question [(4)]. Secondly, we are concerned with forbidden β-decay in both cases; thirdly, we cannot neglect the Coulomb effect for Z-values as high as 37, corresponding to the Rubidium recoil. Without any exaggeration we may state that it would be of much more theoretical interest to examine the recoil from the lightest noble gases, e. g. He^6 and Ne^{23}. In these cases, however, we can only measure the recoils by the current they produce in the condenser. The intensities which are available give rise to very small currents.

The positive current going to the positive plate in the average energy condenser will be of the order of magnitude

$$i = \frac{\langle E_R \rangle}{e V} \cdot 10^{-12} \text{ ampere/millicurie.} \tag{9}$$

The construction of the so-called vibrating reed electrometers [(5)] has made it possible to detect currents as low as 10^{-17} ampere, and it should thus be possible to carry out a measurement with a few microcuries of a radioactive gas.

Of course we get a current not only of recoil atoms but also of electrons. The positive current going to the positive plate is proportional to the average energy of the recoil particles, and the negative current will be approximately equal to half the number of disintegrations per second times the electronic charge e because the motion of the β-particles will not be influenced to any large extent by the very weak electric field. Consequently we find for a given value of V a current going to the positive plate given by

$$i_+ = Ne \left\{ \langle E_R \rangle / (6\ Ve) - 1/2 \right\}. \tag{10}$$

where N is the number of disintegrations per sec. In this formula we have neglected the current from the secondary electrons emitted during the β-process itself. These electrons usually have a rather low energy i. e. a few eV. Consequently they may be removed by means of a magnetic field parallel to the plates of the condenser. This important point will be discussed in more detail in the following.

Magnetic Fields.

We next consider the motion of the disintegration products in a homogeneous magnetic field. The recoil particles move in

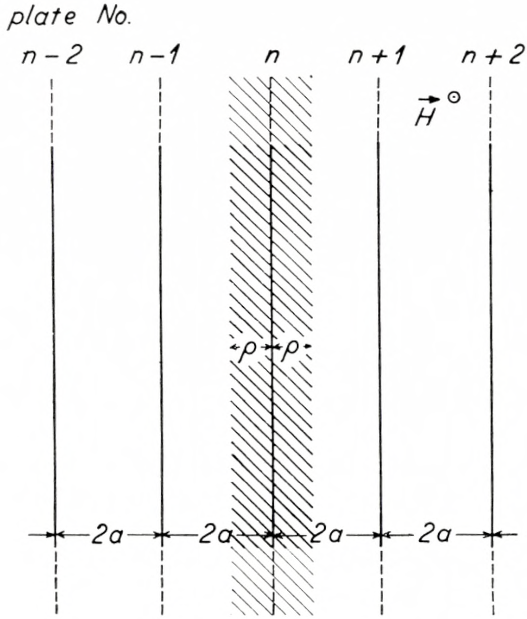

Fig. 4. A series of equidistante planes. $\vec{H}$ is perpendicular to the plane of the figure.

helical orbits, and the radius of the circular motion in a plane perpendicular to the field is

$$\varrho = \frac{p_R \cdot c \cdot \sin\theta}{\mathrm{Hez}}, \tag{11}$$

where θ is the angle between the momentum $\vec{p_R}$ and the magnetic field $\vec{H}$, so that $0 \leqslant \theta \leqslant \pi$.

Suppose that the radioactive gass fills a large space in which is placed a large number of parallel plane plates with a spacing of 2 a (see Fig. 4). The homogenous magnetic field is parallel to the plates. We need only discuss the projection of the particle orbits on a plane perpendicular to the magnetic field. Let us first take all those orbits for which ϱ has the same value. The centres of the circles in which the particles move are evidently distributed homogeneously all over the space. For the particles hitting the plates the distance between the centre and a plate is less than ϱ. Therefore, a fraction $x = \varrho/a$ will hit the plates if $\varrho < a$ and, of course, if $\varrho > a$ all particles hit the plates, or $x = 1$.

Summing over all recoil particles we get a particular simple case if all ϱ's are smaller than a, i. e., if

$$p_R^{\max} \ll \mathrm{Hea}/c, \tag{12}$$

which is a condition similar to (2). When (12) is fulfilled so that $x = \varrho/a$ the fraction N_1/N of the total number of recoils, N, that are able to reach the plates is given by the simple expression

$$\left.\begin{aligned} N_1/N &= \sum_z \int_0^{\pi}\int_0^{p_R^{\max}} \varrho/a \cdot \frac{1}{2} \sin\theta\, d\theta\, P(p_R)\, dp_R S(p_R, z). \\ &= \frac{\pi}{4}\frac{c}{\mathrm{Hea}} \langle p_R/z \rangle. \end{aligned}\right\} \tag{13}$$

In this formula $P(p_R) \cdot S(p_R, z)$ is the relative probability that a particle is emitted with the momentum p_R and the charge value z. We note that usually, to a good approximation, $\langle p_R/z\rangle = \langle p_R\rangle \langle 1/z\rangle$.

It is evident that a measurement on a gas in the space between just two parallel plates (Fig. 5) of this kind is equivalent to a measurement on the whole periodic set of plates. We thus have an apparatus closely similar to that discussed in the previous paragraph, only with the electric field replaced by a magnetic field.

Instead of considering a number of parallel plates we may imagine any kind of tube, parallel to the magnetic field, and with a cross section such that by periodic continuation it can completely fill out the plane perpendicular to $\vec{H}$; examples of this are shown in Fig. 6. We can then again use the argument based on the homogeneous distribution of the centres of the circles, which makes it easy to find the relative number of recoils hitting the walls of a single tube.

As a simple example we take tubes with cross sections of the kind shown in Fig. 6, and which have an inscribed circle, the

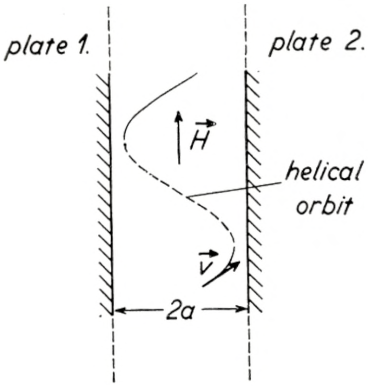

Fig. 5. The collecting system used in the average momentum instrument.

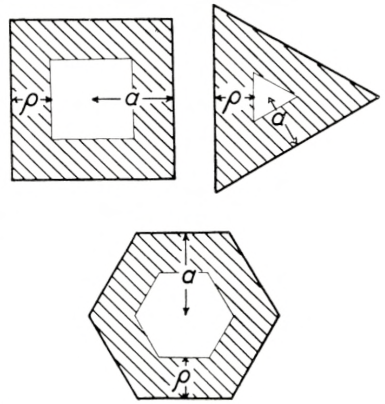

Fig. 6. Sections of tubes of the type where the section has an inscribed circle.

radius of which we take to be a. If the condition (12) is fullfilled we find for all such tubes

$$N_1/N = \frac{\pi}{2} \cdot \frac{c}{\mathrm{Hea}} \langle p_R/z \rangle - \frac{2}{3} \left(\frac{c}{\mathrm{Hea}} \right)^2 \langle p_R^2/z^2 \rangle . \tag{14}$$

It may be remarked that for all tubes allowing their periodic continuation and for the magnetic field fullfilling conditions similar to (12) one can measure a combination of $\langle p_R/z \rangle$ and $\langle p_R^2/z^2 \rangle$ only, like in (14).

The measurement of N_1 can be performed if there is a radioactive daughter substance, as in the measurements mentioned in the description of the average energy instrument. If one wants to use a gas where the daughter substance is not radioactive one can measure directly the electric current to the plates instead of the number reaching the plates. Now, the electrons emitted in the β-decay have momenta of the same order of magnitude as the recoils, and their contribution to the current will compete with that from the recoils. The number of electrons reaching the plates will be given by an expression similar to (13). It is seen that the total current to the two parallel plane plates becomes

$$i = \frac{\pi}{4} Ne \frac{c}{\mathrm{Hea}} (\langle p_R \rangle - \langle p_\beta \rangle), \tag{15}$$

where $\langle p_\beta \rangle$ is the average momentum of the electron. Here one has the advantage that the charge of the recoils does not appear in the formula for the current. The effect from the secondary electrons is quite small and has been neglected (see equ. 28).

Finally, consider the case of the two parallel collector plates when the condition (12) is not fullfilled. For a single line the Number, N_1, of recoils hitting the plates is then found to be

$$N_1/N = \frac{1}{2}\left\{\sqrt{1-\left(\frac{\mathrm{Hea}}{cp_R}\right)^2} + \frac{cp_R}{\mathrm{Hea}} \operatorname{Arc\,sin} \frac{\mathrm{Hea}}{cp_R}\right\}. \tag{16}$$

For low magnetic fields we may write (16) as the following series

$$N_1/N = \left\{1 - 1/6\left(\frac{h}{p_R}\right)^2 - 1/40\left(\frac{h}{p_R}\right)^4 - \cdots\right\}, \tag{17}$$

where

$$h = \frac{\mathrm{Hea}}{c} \tag{18}$$

is a measure of the magnetic field involving the geometrical parameter, a, of the apparatus; h has the dimension of a momentum. The series (17) converges for $(h/p_R) < 1$. However, the convergence is less rapid than might appear from the terms in (17).

The curves for $\frac{1}{2} N_1/N = N_1'/N = n(h)$ represented by (13) and (16) where N_1' is the number of recoils reaching one of the collector plates and for $\left(\frac{8}{\pi} nh\right)$ are illustrated in Fig. 7 as functions of h. For $h > p_R$ the curve $8hn/\pi$ is a constant giving directly the value of p_R. For $h = 0$ these functions equal $1/2$ and 0, respectively.

For a spectrum we find when (12) is not fulfilled,

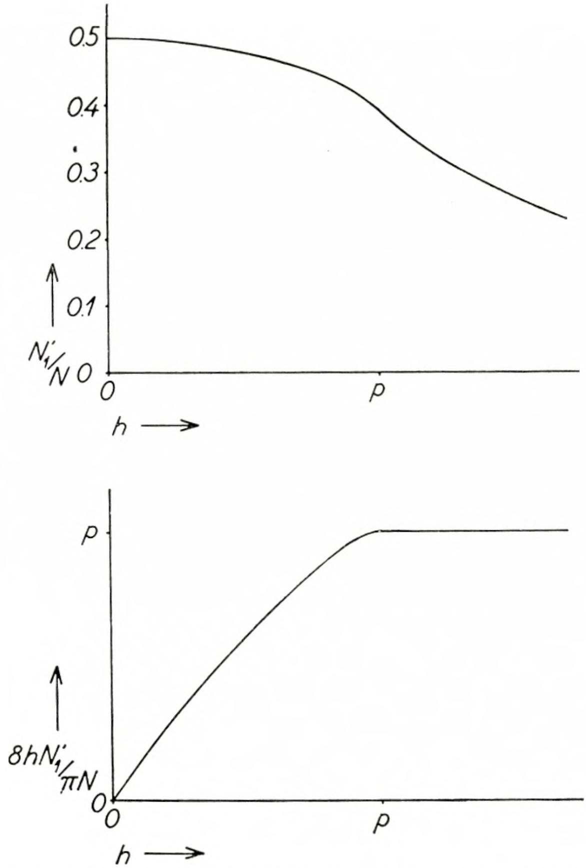

Fig. 7. N_1'/N and $8\,hN_1'/\pi N$ as functions of h for a single line with the momentum p.

$$n(h) = \int_h^{p_R^{max}} \frac{1}{4}\left\{\sqrt{1-\left(\frac{h}{p_R}\right)^2} + \frac{p_R}{h}\,\mathrm{Arc}\sin\frac{h}{p_R}\right\} P(p_R)\,dp_R + \int_0^h \frac{\pi}{8}\frac{p_R}{h} P(p_R)\,dp_R, \tag{19}$$

and in an analysis of a curve $n = n(h)$ in order to obtain the momentum distribution we find the relation

$$P(h) = -12\,h\frac{d^2}{dh^2}n - 4\,h^2\frac{d^3}{dh^3}n, \tag{20}$$

which shows the one-to-one correspondance between n and the momentum distribution.

Discussion of Experimental Questions.

Finite Extension of Collector Plates.

So far we have considered only idealized measuring instruments. We must therefore treat the numerous corrections that can come in so that the effectiveness of the apparatus can be estimated.

An important question will be the size of the instrument. Suppose that one wants to utilize a certain finite region of the

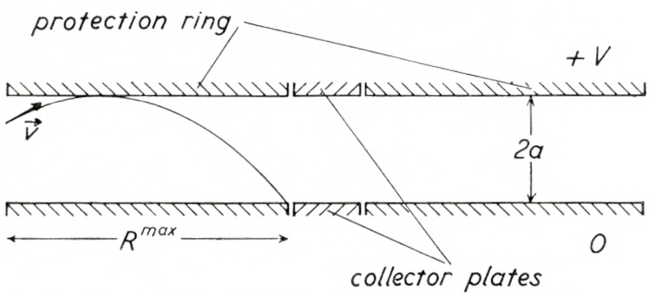

Fig. 8. The figure shows that parabolic orbit which corresponds to the maximum value of R.

condensers or tubes mentioned above. The question is then how far one must extend the condenser plates and the homogeneous field beyond this finite region to ensure that no disturbing end effects occur.

We discuss first the condenser with an electric field. Let R be the projection of the parabolic orbits on the plan of the condenser. In Fig. 8 is shown an orbit and its projection R. Now, to prevent particles created outside the condenser from reaching the collectors, i. e., the central part of the plates we must demand that the protection ring has a width equal to the maximum value of R. R is a maximum when $E_R = E_R^{\max}$ and when V corresponds to the limiting case of equality in (2). Furthermore this parabola shall touch the positive plate of the condenser in Fig. 8. One finds easily

$$R^{\max} = 3\sqrt{3} \cdot a, \tag{21}$$

where 2 a is the distance between the condenser plates.

In the instrument for the measurement of the average momentum it is evident that we want to have extra condenser plates corresponding to complete periods in the periodical motion in the magnetic field. The largest period corresponds of course to a particle with the maximum energy and to the limiting case of the equality in (12). In the direction perpendicular to $\vec{H}$ the

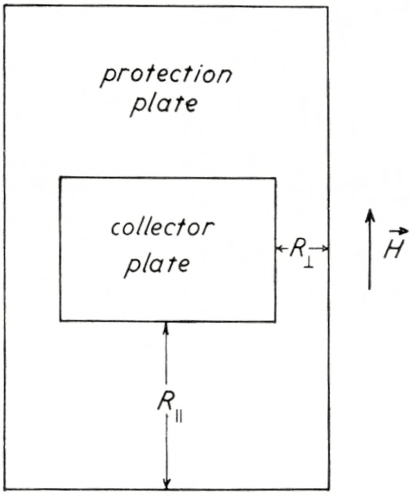

Fig. 9. $R_\perp$ and $R_{||}$ for the magnetic field instrument.

largest period is found in a motion entirely in the plane perpendicular to $\vec{H}$ and its magnitude is given by

$$R_\perp = 2\,a. \tag{22}$$

In the direction parallel to $\vec{H}$ the largest period is found for a particle moving in the direction of the field. Such a particle is not at all influenced by $\vec{H}$ but we may introduce a period as the limiting value of the period for particles moving nearly parallel to $\vec{H}$. In this way we find

$$R_{||} = 2\,\pi\,a. \tag{23}$$

The magnitude of $R_\perp$ and $R_{||}$ is illustrated in Fig. 9. For He^6 and Ne^{23} we have that $p^{\max} \simeq 9$ mc and it follows from the condition (12) that we must have at least $Ha \simeq 15000$ Gauss cm. This shows that it is necessary to have a magnetic field extending over a rather large region of space.

The conditions (21), (22), and (23) hold when the conditions (2) and (12) are fulfilled. If this is not the case the protection areas must be larger. We need not discuss this point in detail but only notice the order of magnitude of the protection ring necessary in the extreme case of $F = H = 0$ when a certain accuracy in the experiment is demanded. Let us furthermore limit ourselves to the case of the collector plate being the central part of a circular plate condenser with the radius R. It can easily be shown that to the first order in a/R the number of particles reaching one of the collectors is given by

$$N_1'/N = \frac{1}{2}(1 - a/R). \tag{24}$$

This means that the protection ring must be rather large to enable us to apply formulas like (3) down to very low voltages. However, the effect (24) diminishes rapidly for increasing electric potential difference V in the condenser.

Field Inhomogenities.

So far, we have treated homogeneous fields only. However, it is clear that deviations from the desired homogenity may occur. Therefore, we shall discuss qualitatively the influence of field inhomogenities in order to find an upper limit to the inhomogenities when a certain accuracy in the experiment is demanded.

Consider first the case of a nearly homogeneous electric field. Inhomogenities can arise from the effects of the ends of the condenser. As an example one may take two circular plates of radius R and placed at a distance $2\,a$. We shall be interested in the case $R \gg 2\,a$, for which case a simple solution has been given by Rose [(3)]. The inhomogenities near the border will then decrease with the distance from the edge, and be proportional to $\exp\left\{-\pi(R-r)/a\right\}$. Because of this rapid exponential decrease the field can be closely homogeneous in the greater part of the condenser. This shows that effects from the edges easily can be made sufficiently small. However, small deviations from the desired form of the surface of the condenser plates or variations in the contact potential of the surface due to impurities may cause local inhomogenities of the field.

One can make a rough estimate of the change in N_+ and N_- which results from inhomogenities of the above kind. Let us consider a rather extended inhomogenity of the field, e. g.,

$$\left.\begin{aligned} \varphi &= -F_0 z - b\left(z^2 - \frac{1}{2} r^2\right) \\ F(z, r) &= (-bx, -by, F_0 + 2\,bz) \end{aligned}\right\} \tag{25}$$

where z is the distance across the condenser as measured from the central plane, and r is the distance from the axis. We assume that $ba \ll F_0$. With a field of this kind one finds that in the central region of the condenser

$$N_+/N = \langle \frac{1}{6} \frac{E_R}{eV} \rangle \left(1 + C \frac{ba^2}{V}\right), \tag{26}$$

where C is of the order of unity. Therefore, ba^2/V is a measure of the relative change in the number of particles striking the positive plate.

In the instrument with a magnetic field the influence of inhomogenities will be of a similar order of magnitude as for electric fields. But it should be noted that in addition special effects may come in. For instance, if the gradient of the magnetic field has a component perpendicular to $\vec{H}$ and parallel to the collector plates the circles in which the particles move will not only travel in the direction of the field but also have a motion perpendicular to the plates. There can therefore be a slight tendency for the particles to move towards one of the plates. An estimate of this effect shows, however, that the relative correction to N_1 is less than $\Delta H/H$, where ΔH is the total change of H through the instrument.

In this connection we note that it is important that the magnetic field is exactly parallel to the collector plates. A deviation by an angle α will bring about a relative error in the measurement of about $\alpha l/(4\,\mathrm{a}) \simeq \alpha$ where l is the dimension of the instrument parallel to $\vec{H}$.

We found that field inhomogenities do not give rise to relative errors larger than the inhomogenities themselves. Apart from inhomogenities in the field there may be differences in the density in space of the radioactive substance if one is concerned with a very short lived isotope. Effects of this kind have been studied in the examination of the Kr^{89} recoil [(2)].

Secondary Electrons, Collisions.

In case we want to measure the currents produced by the recoils and the β-particles the most important source of error will arise from the secondary electrons. These electrons may either be ejected from the recoil ion immediately after the emission of the β-particle or by internal conversion of subsequent γ-rays, or they can be secondary electrons from the walls of the chamber and from collisions with gas atoms. In either case they are of comparatively low energy.

Let us consider the electric field instrument first. In order to eliminate the secondary electrons it will be necessary to apply a weak magnetic field parallel to the condenser plates. Accordingly, we are lead to discuss the motion of charged particles in crossed electric and magnetic fields so as to find corrections to be applied to our formula (10) due to the influence of a magnetic field on the motion of the recoils and due to the combined action of the electric and magnetic fields on the motion of the β-particles. This means that the contribution from the electrons to the current (10) may deviate considerably from $\frac{1}{2}Ne$; instead we may put in (10) a function of H, F and—as neither (2) nor (12) is fullfilled for the electrons—of the geometry of the instrument as mentioned in connection with (24); i. e., we put

$$i_{\beta+} = Ne \cdot K_G(F, H), \tag{27}$$

where K is a certain function of F and H and where the index G refers to the geometry of the instrument. It can be shown that the influence of the electric field may be neglected within certain limits and consequently we may omit the variable F in the function K in (27).

In the average momentum instrument the effect from the secondary electrons is small. The secondary electrons from the recoil atoms will give a relative increase in the right hand side of equation (15) of order of magnitude

$$x = \frac{n}{N}\frac{\langle p_e\rangle}{\langle p_\beta\rangle} = (\langle z\rangle - 1)\frac{\langle p_e\rangle}{\langle p_\beta\rangle}, \tag{28}$$

where n is the number of secondaries, $\langle p_e\rangle$ their average momentum and $\langle z\rangle$ is the average charge of the recoil ions. In this expression for x, the first factor n/N can be considerably smaller than unity, e. g., about $^1/_{10}$. The average momentum of the secondaries will usually correspond to an energy not larger than 100 eV. If $\langle p_\beta\rangle$ corresponds to an energy of more than one MeV the ratio $\langle p_e\rangle/\langle p_\beta\rangle$ will be less than $^1/_{100}$. Therefore x can easily be less than 1 $^0/_{00}$. This illustrates the order of magnitude of x. Of course, for each particular β-emitter x can be estimated rather accurately. In unfavourable cases x may be considerably larger than 1 $^0/_{00}$; for if the β-decay is followed by γ-rays with a large internal conversion coefficient the number of electrons ejected per recoil ion, or n/N, can become somewhat larger than 1 because of Auger effect. For this reason also it is especially convenient that $n/N = \langle z\rangle - 1$ can be determined directly from the experiment, cf. equ. (30).

Inhomogenities in the field may allow some of the secondary electrons emitted from the collectors to move away along the lines of force. This effect disappears if the inhomogenities are symmetrical about the centre of the instrument.

Connected with secondary electrons is the question of change of charge of the recoils during their flight. Thus, the emission of the β-particle may give rise to a release of one or more electrons bound in the recoil atom. The life-times for emission of these secondaries will be of the order of Auger life-times, i. e., less than 10^{-10} sec. However, during this time interval the recoils can only travel about 10^{-3} cm, and therefore the secondaries can be regarded as being emitted immediately after the β-decay. It is seen that the charge values that a negative β-emitter can take on are $+1$, 2, etc. while a positive β-emitter can have $z = -1, 0, +1$, etc. The possibility of charge values 0 and -1 makes measurements of the present kind on positive β-emitters slightly different from the case of negative β-emitters.

Collisions between the recoils and the residual gas atoms can cause a change of charge but will always lead to energy loss. It is therefore essential that the pressure is sufficiently low. One will expect that, denoting the mean free path by λ and the length of the collector plates by l, the relative effect of collisions will be of the kind $\sim l/\lambda \cdot \log \lambda/l$ and $\sim a/\lambda$ for the magnetic and electric field instruments, respectively. The magnitude of the effect can be measured directly by applying different pressures, so that this source of error can be estimated accurately by experiment. The experiments on the recoil of Kr^{89} mentioned above indicate that the pressure effect is $\sim 1\ {}^0/_{00}$ at pressures of about $2 \cdot 10^{-5}$ mm Hg for the more unfavourable case, the magnetic field instrument.

The Determination of N.

In order to determine average values of energy or momentum of the kind discussed in the two first sections it will be necessary to know N, the total number of disintegrations. In the average energy instrument a measure of N is easily found when the number of particles can be counted, i. e., when the radioactive noble gas has a radioactive daughter substance. In this case $N = N_+ + N_-$ where N_- and N_+ are the number of particles collected on the negative and the positive plate respectively. But when currents are measured N must be determined in a different manner. According to (10) and (27) we get

$$i_+ = Ne\left\{1/6\,\frac{\langle E_R\rangle}{Ve} - \mathrm{K}_G(H)\right\} \tag{29}$$

in that approximation where the influence of the electric field on the β-particles and of the magnetic field on the recoils can be neglected. In the same approximation i_- is given by

$$i_- = Ne\left\{\langle z\rangle - \frac{1}{6}\,\frac{\langle E_R\rangle}{Ve} - K_G(H)\right\}. \tag{30}$$

Here, one further unknown quantity, $\langle z \rangle$, comes in which complicates the determination of N. A measurement at two different voltages gives

$$\langle E_R \rangle = 6\, V_1 V_2 K_G (H) \frac{x - 1}{x V_2 - V_1}, \tag{31}$$

where $x = i_2/i_1$. It is of importance that we cannot hope to determine $\langle E_R \rangle$ with greater accuracy than K or N can be determined. (30) further allows $\langle z \rangle$ to be determined and with the same accuracy as N can be measured. The determination of N is complicated and usually can not be carried through with any great precission.

In the magnetic field case we meet with the same difficulty and with the further complication that even when the number of recoils reaching the collectors are counted the determination of N must be carried out separately before (15) can be used. This difficulty can be overcome by first measuring i according to (15) and after that applying an electric field across the condenser so strong (a few thousand volts) that no recoils can reach the positive collector plate but at the same time weak enough so as not to affect the electrons. Neglecting some corrections which we shall calculate in the following sections the current is then given by the second term in (15). By measuring the currents with and without electric field we find

$$\left| i/i_+ \right| = \frac{\langle p_R - p_\beta \rangle}{\langle p_\beta \rangle}. \tag{32}$$

The determination of (32) has the advantage that if the measurements are carried out in a double condenser and the magnetic field is the same in the two condensers, this field need not be very accurately constant in time because H is then eliminated in (32).

We notice that the measurement (32) can be carried through with the same accuracy as the average energy determination by means of much less accurate measurements of the currents in question. Further, it is seen that the determination of (32) even for the same accuracy as the determination of the average energy contains more valuable information because (32) gives better

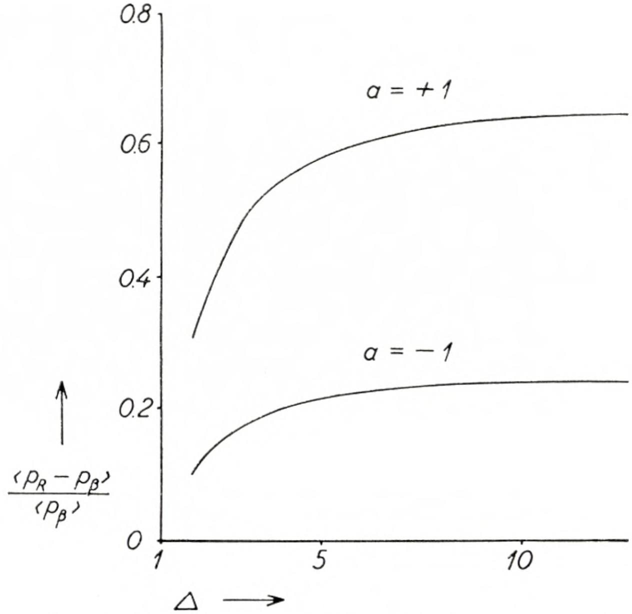

Fig. 10. Theoretical values for $\langle p_R - p_\beta \rangle / \langle p_\beta \rangle$ for different maximum energies and for different angular correlations for the angle between the momenta of the electrom and the neutrino.

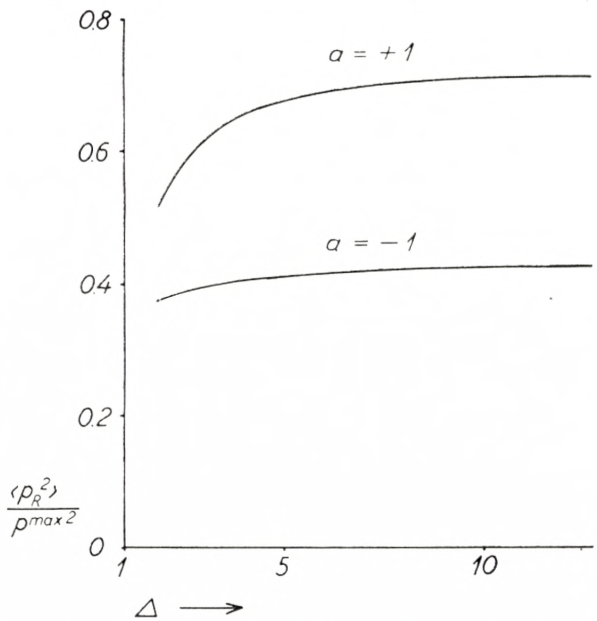

Fig. 11. Theoretical values for $\langle p_R^2 \rangle / p_R^{\max 2}$ for different maximum energies and for different angular correlations as in Fig. 10.

possibilities for distinguishing between the different angular correlation between electron and neutrino in β-decay. This correlation is of the type $(1 + av_\beta/c \cdot \cos\theta_{\beta\nu})$. Fig. 10 shows (32) for different maximum β-energies and for different values for a in this correlation function. Fig. 11 gives $\langle E_R \rangle / E_R^{\max}$ in the same kind of plot[(6)]. It is seen that the relative difference between the curves for the different possibilities for a is more pronounced in Fig. 10 than in Fig. 11.

The above discussion and in particular the application of equation (32) leads us to study in more detail the motion of particles in crossed electric and magnetic fields so as to find the corrections to be applied and to determine the region of validity of the approximations used in formulas (29), (30) and (32).

Crossed Electric and Magnetic Fields.

The Equation of Motion.

We shall now treat the relativistic motion of charged particles in homogeneous, perpendicular electric and magnetic fields. Let the electrical field point in the direction of the x-axis, with the numerical value F, while the magnetic field has the direction of the z-axis and the value H. We will be interested primarily in the motion in the x-direction, and the collector plates will be parallel to the y, z-plane. The motion is governed by

$$\frac{d}{dt}\vec{p} = e\vec{F} + \frac{e}{c}\left[\vec{\dot{r}}, \vec{H}\right], \tag{33}$$

$\vec{p}$ being the momentum and $\vec{\dot{r}}$ the velocity of the particle. Conservation of energy gives

$$\frac{mc^2}{\sqrt{1-\left(\vec{\dot{r}}\right)^2/c^2}} = Fe\,(x + x_0) + \frac{mc^2}{\sqrt{1 - v_0^2/c^2}}, \tag{34}$$

where m is the rest mass of the particle, v_0 the initial velocity and x_0 the initial value of x.

Introducing (34) in the x, y-components of (33) and eliminating y from the resulting equations we find an equation for the motion in the x-direction,

$$\ddot{s}s^2 + \dot{s}^2 s - (c^2 - \beta^2)\, s - \alpha\beta\,(\beta + v_{0y}) = 0, \tag{35}$$

with

$$\beta = cH/F,$$

$$\alpha = \frac{mc^2}{\sqrt{1 - v_0^2/c^2}} \cdot \frac{1}{eF},$$

and

$$s = x - x_0 + \alpha;$$

β has the dimension of a velocity, while α and s are lengths.

Integrating (35) once we find

$$s^2\dot{s}^2 = As^2 + 2\,Bs + C, \tag{36}$$

with

$$A = c^2 - \beta^2,$$

$$B = \alpha\beta\,(\beta + v_0),$$

and

$$C = \alpha^2\,(v_{0x}{}^2 - \beta^2 - 2\,\beta v_{0y} - c^2).$$

From this s, and x, can be found as a function of t. However, in the present connection we are looking only for possible minimum and maximum values of x, in order to find the maximum distance for collection of particles on the plates. We therefore put $\dot{s} = 0$ in (36), and the solution of the resulting equation

$$f(s) = As^2 + 2\,Bs + C = 0, \tag{37}$$

will give the extrema of s from which $x^{\max}$ and $x^{\min}$ can be found. Since $B^2 \geqslant AC$ the equation has always real solutions. According to (36) the orbits obey the inequality $f(s) \geqslant 0$. Therefore, for $A < 0$, i.e., F < H, the solutions of (37) will represent the minimum and the maximum of a periodic motion in x of the particle. Since for $A < 0$ the magnetic field is stronger than the electric field the motion is just in this case dominated by the magnetic field. Though the condition, $H > F$, is independent of the mass, charge, and initial velocity of the particle the period in x will depend on these quantities. For $A > 0$ the motion described by (36) will lie outside the solutions of (37), and if the particle is positively charged the motion in x will have a minimum but not a maximum. In this case, where $F > H$, the motion is thus aperiodic and resembles the motion in a pure electric field.

Now, since we want to have a periodic motion in x, at least for the secondary electrons in order that they can move away along the magnetic lines of force, we can limit ourselves to the case of $A < 0$, or $F < H$. Inserting the values for α, β and s we find

$$x^{\substack{\max\\ \min}} - x_0 = \frac{mc^2}{e\sqrt{1 - v_0^2/c^2}} \cdot \frac{F}{H^2 - F^2} \cdot$$

$$\left\{1 + \frac{H}{F}\frac{v_{0y}}{c} \pm \sqrt{\left(1 + \frac{H}{F}\frac{v_{0y}}{c}\right)^2 + \frac{v_{0x}^2}{c^2}\left(\frac{H^2}{F^2} - 1\right)}\right\}. \qquad (38)$$

We are particularly interested in the oscillation length, $x^{\max} - x^{\min}$. Denoting this quantity by $2\,l$ we find from (38) that

$$l = \frac{mc^2}{e\sqrt{1 - v_0^2/c^2}} \cdot \frac{F}{H^2 - F^2}\sqrt{\left(1 + \frac{H}{F}\frac{v_{0y}}{c}\right)^2 + \frac{v_{0x}^2}{c^2}\left(\frac{H^2}{F^2} - 1\right)}. \qquad (39)$$

It is seen that for F/H approaching 1, l tends to infinity.

For completeness we shall write down the expression corresponding to (38) as found in a non-relativistic calculation. Here,

$$x^{\substack{\max\\ \min}} - x_0 = mc^2 \frac{F}{H^2}\left\{1 + \frac{H}{F}\frac{v_{0y}}{c} \pm \sqrt{\left(1 + \frac{H}{F}\frac{v_{0y}}{c}\right)^2 + \frac{v_{0y}^2}{c^2}\frac{H^2}{F^2}}\right\}.$$

(non-rel.) (40)

The equation (40) differs from the relativistic expression when F/H is close to unity. This deviation is connected with the fact that in the non-relativistic approximation the orbits will be periodic in x for any values of F and H.

With the use of (38), (39), and (40) we can now proceed to calculate the number of particles striking the collector plates.

Number of Particles Hitting the Collector Plates.

From the number of particles emitted in different directions from every space point we can with the use of (38) calculate the total number of particles hitting the collector plates. The number of particles emitted between x_0 and $x_0 + dx_0$, in the angular

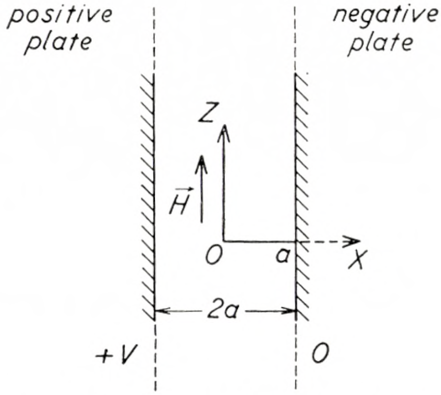

Fig. 12. The collector system in the instrument utilizing crossed electric and magnetic fields.

intervals $d\varphi$, $d\theta$ and with energy between E and $E + dE$, and charge value z, will be given by

$$N \cdot \frac{dx_0}{2a} \cdot \frac{1}{4\pi} \sin\theta \, d\theta \, d\varphi \, P(E) \, S(E, z) \, dE. \tag{41}$$

In this formula $P(E)\ S(E, z)$ is the relative probability that a particle is emitted with the energy E and the charge z. The direction of the fields is as in the preceding paragraph, and the positive and negative condenser plates are the planes $x = -a$ and $x = a$, respectively (see Fig. 12).

As in the case of single fields we may first consider some particularly simple cases. It is important that for given values of F and H the recoil particles and the electrons behave quite differently. In particular the large mass of the recoil particles means that according to (39) their oscillation length in the x-direction is more than 10^3 times that of the electrons.

Let us demand that the orbits of all recoils have $x^{\max} > a$. This means that particles travelling towards the negative plate are not able to go back to the positive plate but will strike the negative plate. This simplifies the discussion considerably. The condition for this is evidently that the right hand side of (38) shall remain greater than $2\,a$ if we use the $+$ sign before the square root and choose the most unfavourable value of v_{0y}. This leads to

$$a < \frac{Mc^2 F}{eH^2}\left(1 - \frac{H}{F}\frac{v_R^{\max}}{c}\right). \tag{42}$$

Now, the motion of the β-particles is mainly governed by the magnetic field and therefore, as mentioned above, it is favourable to have magnetic fields of the order $eHa \sim p^{\max}/c$. Inserting this in (42) we find an approximate lower limit on F

$$F \gtrsim 2\,Hv_R^{\max}/c. \tag{43}$$

The number of particles striking the positive collector plate can be calculated in the following manner. For a given value of $\vec{v}_0$ we first find the upper limit of the starting point x_0, let us call it x^{up}, for which the particles hit the positive plate. We therefore put $x^{\min} = -a$ in equation (38) and the left hand side becomes equal to $-(a + x^{\mathrm{up}})$; in the right hand side we use the minus sign in front of the square root because we are concerned with $x^{\min}$. Since (43) shows that $(H/F)\,(v_R^{\max}/c)$ is smaller than unity, we develop the square root in powers of this quantity. Finally, using (41) we sum over all velocities of emission and over the direction of emission remembering that the summation only goes over those particles for which the initial motion is towards the positive plate, when (43) is fullfilled. The total current of recoils to the positive plate is then found to be

$$i_{+R} = \frac{Ne}{V}\cdot\frac{1}{6}\left[\langle E_R\rangle + \frac{1}{20}\left(\frac{H}{F}\frac{v_R^{\max}}{c}\right)^2\cdot\langle E_R\frac{v_R^2}{v_R^{\max^2}}\rangle + \cdots\right]. \tag{44}$$

This gives immediately the current of recoils to the negative plate

$$i_{-R} = Ne\,\langle z\rangle - i_{+R}. \tag{45}$$

As to the β-particles, the influence of the electric field will because of their high energy remain only a small perturbation on their motion in the magnetic field. For these particles we can again demand that the period $2l$ of their oscillatory motion in the x-direction is always smaller than $2a$. Using the value of l in (39), and with F smaller than H, we thus have the condition

$$l^{\max} = \frac{F W_\beta^{\max}}{e H^2}\left(1 + \frac{H}{F}\frac{v^{\max}}{c}\right) = \frac{c p^{\max}}{H e} + \frac{W_\beta^{\max}}{H e}\left(\frac{F}{H}\right) < a, \tag{46}$$

where W_β is the energy of the β-particle including the rest mass.

For pure magnetic fields the condition was, equ. (12), $cp^{\max}/(He) \ll a$. The extra term $(W_\beta^{\max}/He)(F/H)$ in (46) is small for relativistic particles (it can be e. g. about $1\,^0/_{00}$) if F is small compared with H, and in the usual cases (46) reduces to (12).

The two conditions (43) and (46) can be combined in a simple manner. Neglecting the small term in (46) we get a lower limit on Ha and introducing this in (43) we find, since $V = 2aF$,

$$eV > eV' = 8\,E_R^{\max}, \tag{47}$$

which shows the lower limit on the potential difference that is to be imposed in order that the present simple formulae exhibiting a marked difference between β-particles and recoils can be applied. For the lightest noble gases, He^6, Ne^{23}, and A^{41} we find for V' the values 12000, 4000, and 400 volts, respectively.

In the calculation of the current $i_{+\beta}$ of β-particles to the positive plate we can make a series development of the square root in (38), developing in powers of $(F/H)(c/v)$. It is here convenient to write

$$\begin{aligned}\sqrt{\left(1+\frac{Hv_y}{Fc}\right)^2 + \frac{v_x^2}{c^2}\left(\frac{H^2}{F^2}-1\right)} &\simeq \left(1+\frac{Hv_\perp}{Fc}\right)\sqrt{1 - 2\,\frac{H(v_\perp - v_y)}{Fc\left(1+\dfrac{Hv_\perp}{Fc}\right)^2}} \\ &\simeq 1 + \frac{H}{F}\frac{v_\perp}{c} - \frac{v_\perp - v_y}{v_\perp} + \cdots,\end{aligned} \tag{48}$$

where $v_\perp^2 = v_x^2 + v_y^2$, and where we have neglected higher order terms in F. By integration over the angles and the velocities the final expression for the current becomes

$$i_{\pm\beta} = -Ne\left[\frac{\pi}{2}\frac{c\langle p_\beta\rangle}{2\,aHe} \pm \frac{\langle W_\beta\rangle}{2\,aHe}\frac{F}{H} + \cdots\right], \tag{49}$$

where higher terms in F/H are neglected. In pure magnetic fields we have only the first term inside the brackets. The change due to the electric field is in relative measure approximately the same in (49) as in (46). The formulae (44), (45), and (49) give together the correct expressions for the current to the plates. The total currents are in the first approximation

$$i_+ = Ne\left(\frac{1}{6}\frac{\langle E_R\rangle}{2\,aFe} - \frac{\pi}{4}\frac{\langle cp_\beta\rangle}{2\,aHe} - \cdots\right), \tag{50}$$

$$i_- = Ne\left(\langle z\rangle - \frac{1}{6}\frac{\langle E_R\rangle}{2\,aFe} - \frac{\pi}{4}\frac{\langle cp_\beta\rangle}{2\,aHe} + \cdots\right). \tag{51}$$

It is thus seen that the following quantities can be measured:

$$\frac{\langle p_\beta - p_R\rangle}{\langle p_\beta\rangle} \text{ from (15)}, \quad \langle E_R\rangle/\langle p_\beta\rangle\,c, \quad \langle z_R\rangle/\langle p_\beta\rangle$$

and by an absolute calibration of the instrument we can find $N\langle p_\beta\rangle$ so that the total number of disintegrations can be determined with the accuracy with which $\langle p_\beta\rangle$ is known from experiment or theory. The $\langle E_R\rangle$-term in (50) or (51) is of cource found by measuring $i_\pm$ for different values of F. The accuracy of the determination of $\langle E_R\rangle/\langle p_\beta\rangle c$ is much less favourable than for the other quantities since the $\langle E_R\rangle$-term in these formulae is small compared with the $\langle p_\beta\rangle$-term. When comparing the measurement of $\langle E_R\rangle$ in (50) and (51) with that in the electric field instrument, (29), one finds that the accuracy that can be obtained is of the same order of magnitude in the two cases. Although in (50), (51) the relative magnitude of the $\langle E_R\rangle$-term in the currents is $\sim$ 3 times less than can be obtained in the electric field instrument the absolute measurements and the knowledge of the geometry of the instrument can be more precise for the crossed field instrument.

Summary.

A discussion is given of some proposed experiments in which average values of the energy and momentum of recoil particles from one atomic gases can be determined. The method is based on a simple connection between these average values and the number of recoils collected on the plates of a plane parallel condenser filled with a radioactive inert gas at a low pressure. The average energy of the recoil particles can be measured for suitable electric potential differences between the plates. If instead one uses a magnetic field parallel to the plates the average momenta are obtained. There are significant differences between measuring the number of recoils striking the plates and the currents to the plates.

In order to find the effectiveness of the instrument a number of possible deviations from the idealized instrument are treated. These include the finite size of the condenser, inhomogenities of the field, the question of secondary electrons, and difficulties in the determination of the total number of disintegrations.

The discussion shows that it is most favourable to compare measurements of currents performed first with a purely magnetic field and secondly with a combined electric and magnetic field. In this way it is possible to determine the difference between the average values of the recoil and β-particle momenta divided by the average value of the momentum of the β-particles. This ratio depends strongly on the angular correlation in the emission of the β-particle and the neutrino. Furthermore this instrument enables one to find the average value of the recoil energy and the average value of the charge of the recoil ions.

The accuracy with which the pertinent quantities can be

determined in the measurements described here is expected to exeed that of previous recoil experiments. Thus one can hope to obtain a rather precise check on the coupling between the particles involved in β-decay.

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